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**TWO SPACE SCATTERER FORMALISM CALCULATION
OF BULK PARAMETERS OF THUNDERCLOUDS**

Prepared by:
Academic Rank:

Dieudonné D. Phanord, Ph.D.
Associate Professor

Institution and
Department:

University of Wisconsin at Whitewater
Department of Mathematics
and Computer Science

NASA / MSFC:
Laboratory:
Division:
Branch:

Space Science
Earth Sciences / Applications
Remote Sensing

MSFC Colleagues:

William Koshak
Richard Blakeslee
Hugh Christian

I. INTRODUCTION

In (1), the single space scatterer formalism based on (2) is used to evaluate the bulk parameters of a cloud illuminated from outside. The single space scatterer formalism allows an obstacle excited by an incident wave traveling in free space to radiate in free space. The calculations of (1) took only into account contributions due to dry air and water. The approximations for the attenuation agree closely with those obtained using number density and the scattering cross section of water drops. The results are obtained directly and they represent a solid starting point for subsequent developments.

In (3), we used a modified two-space scatterer formalism of Twersky (4) and (5) to establish for a cloud modeled as a statistically homogeneous distribution of spherical water droplets, the dispersion relations that determine its bulk propagation numbers K_j and bulk indexes of refraction η_j ($j = 1, 2$) in terms of the vector equivalent scattering amplitude \vec{G} and the dyadic scattering amplitude \tilde{g} of the single water droplet in isolation. The results of (3) were specialized to the forward direction of scattering while demanding that the scatterers preserve in the sense of (5), the incident polarization. This requirement did allow us in (3) to drop the subscript j and to look for K and η .

Here, we apply (3) to obtain specific numerical values for the macroscopic parameters of the cloud. We work as in (6), with a cloud of density $\rho = 100 \text{ cm}^{-3}$, a wavelength $\lambda = 0.7774 \mu\text{m}$, and with spherical water droplets of common radius $a = 10 \mu\text{m}$. In addition, the scattering medium is divided into three parts, the medium outside the cloud with propagation parameter $k_o = \frac{2\pi}{\lambda} = 8.0823 \mu\text{m}^{-1}$, moist air (the medium inside the cloud but outside the droplets) corresponding to $\kappa_1 = k_o \eta_1$, and the medium inside the spherical water droplets specified by $\kappa_2 = \kappa_1 \eta_2$. The physical constants η_1 , and $\eta_2 = 1.33 + i7.33 \times 10^{-8}$ (from (7)) are the relative indexes of refraction of moist air and water respectively. The numerical values of K and η are important to the innovative work of (8) for lightning due to point sources inside the cloud.

In this modified two-space scatterer formalism, a single spherical water droplet is allowed to be excited by a wave traveling in K -space (the space of the equivalent medium) but to radiate in κ_1 -space instead of the usual free or k_o -space. This is possible since we have a sparse distribution of scatterers (i.e., the particles in the distribution are located at a significant distance away from each other as compared to their radius) and the boundary correction terms due to back and forth reflections from κ_1 -space to k_o -space or vice versa are negligible.

The results of this report are applicable to a cloud of any geometry since the boundary does not interfere with the calculations. Also, it is important to notice the plane wave nature of the incidence wave $\vec{\phi}_1$ in the moist atmosphere. Should the cloud be of a geometry such that its direct response to an initial outside excitation $\vec{\phi}_o$ be of spherical type, one would have to make the plane wave approximation in order to apply the work of Twersky. The theory of Twersky is mostly suitable for a slab distribution of scatterers. However, it can be extended to cover the present problem without a loss of generality since the contributions of the boundary layer surfaces are small and most lightning activities usually occur inside the cloud.

In order to avoid repetition, we cite key equations of (3-5, 10). In general, we work in spherical coordinates. We use bold face or an arrow to denote a vector or a vector operator. A circumflex indicates a vector of unit magnitude. A tilde on the top of a letter denotes a dyadic (second rank tensor). For brevity, we use [5:4] for equation 4 of Re. (5) etc.

II. MATHEMATICAL ANALYSIS

For an incident plane harmonic electromagnetic wave $\vec{\phi}_1 = \hat{\mathbf{a}}_1 e^{i(\vec{\kappa}_1 \cdot \mathbf{r} - i\omega t)}$ in the moist air due to an initial outside excitation $\vec{\phi}_0$ of the cloud, the self-consistent integral equation governing the multiple configurational scattering amplitude as in (5) is

$$\mathbf{G}_t(\hat{\mathbf{r}}) = \tilde{\mathbf{g}}_t(\hat{\mathbf{r}}, \hat{\kappa}_1) \cdot \hat{\mathbf{a}}_1 e^{i\vec{\kappa}_1 \cdot \mathbf{r}_t} + \sum_c' \int \tilde{\mathbf{g}}_t(\hat{\mathbf{r}}, \hat{\mathbf{r}}_c) \cdot \mathbf{G}_m(\hat{\mathbf{r}}_c) e^{i\vec{\kappa}_{1c} \cdot \mathbf{R}_{tm}}, \quad [1]$$

where $\mathbf{R}_{tm} = \mathbf{r}_t - \mathbf{r}_m$, $\int_c = \frac{1}{2\pi} \int d\Omega_c$. The single dyadic scattering amplitude $\tilde{\mathbf{g}}(\hat{\mathbf{r}}, \hat{\kappa}_1) \cdot \hat{\mathbf{a}}_1 = \mathbf{g}(\hat{\mathbf{r}}, \hat{\kappa}_1 : \hat{\mathbf{a}}_1)$ is defined in (3:4). The magnitude of the separation distance $|\mathbf{R}_{tm}| < D$ (the diameter of the cloud).

After taking the ensemble average of [1], using the quasi-crystalline approximation of Lax (9), the equivalent medium approach, and Green's theorems, the dispersion relations determining the bulk parameters (for more details see (5)) are

$$\vec{\mathcal{G}}(\vec{\kappa}_1 | \vec{K}_j) = -\frac{\rho}{c_0(K_j^2 - \kappa_1^2)} \left\{ \left[e^{-i\vec{K}_j \cdot \mathbf{R}}, \vec{U}_j \right] \right\} + \rho \int_{V_D - v} [f(\mathbf{R}) - 1] e^{-i\vec{K}_j \cdot \mathbf{R}} \vec{U}_j d\mathbf{R}, \quad [2]$$

where $d\mathbf{R}$ denotes volume integration over $(V_D - v)$. Here, $\vec{\mathcal{G}}$ is the equivalent scattering amplitude and \vec{U}_j is a radiative function defined by $\vec{U}_j = \int_c \tilde{\mathbf{g}}(\hat{\mathbf{r}}, \hat{\mathbf{r}}_c) \cdot \vec{\mathcal{G}}(\vec{\kappa}_{1c} | \vec{K}_j) e^{i\vec{\kappa}_{1c} \cdot \mathbf{R}}$, and $c_0 = \kappa_1/4\pi i$. The bulk propagation parameter is $K_j = \kappa_1 \eta_j$ with η_j being the bulk index of refraction, and $\{[f, g]\} = \int_S [f \partial_n g - g \partial_n f] dS$ is the Green surface operator. The ensemble is specified as in (5) by the average number ρ of scatterers in unit volume and by $\rho f(\mathbf{R})$ with $f(\mathbf{R})$ as the distribution function for separation of \mathbf{R} pairs.

Similar to (3), the model is required to neglect all phase transition effects (10) and to take only into account pair interaction due to central forces. Neglecting the inter-droplet potential, the distribution function $f(\mathbf{R})$ can be chosen to be always equal to unity. Hence, [2] is simplified to

$$\left[(K^2 - \kappa_1^2) \tilde{\mathbf{L}}_* + \left(\frac{\rho}{c_0} \right) \tilde{\mathbf{g}}(\hat{\mathbf{r}}, \hat{\mathbf{K}}) \right] \cdot \vec{\mathcal{G}}(\vec{\kappa}_1 | \vec{K}) = 0. \quad [3]$$

In [3], we let $\hat{\mathbf{r}} = \hat{\mathbf{K}}$ and use the fact that the scatterers preserve the incident polarization to transform [3] into

$$(K^2 - \kappa_1^2) = -\left(\frac{\rho 4\pi i}{\kappa_1} \right) g(\hat{\mathbf{K}}, \hat{\mathbf{K}}), \quad \eta^2 - 1 = -\left(\frac{\rho 4\pi i}{\kappa_1^3} \right) g(\hat{\mathbf{K}}, \hat{\mathbf{K}}) \quad [4]$$

where the subscript j is no longer necessary. In [4], the two-space scatterer formalism scattering amplitude corresponding to the spherical droplet, a large tenuous scatterer, must be evaluated in the forward direction of scattering (i.e., $g = g(\hat{r}, \hat{r}) = \hat{a}_1 \cdot g(\hat{K}, \hat{K})$). The unit vector \hat{K} , the direction of the bulk propagation vector \vec{K} , is such that $\hat{K} \cdot \hat{a}_1 = 0$.

III. NUMERICAL CALCULATIONS

To start the numerical calculations, we use the WKB approximation corresponding to a bulk excitation associated with an incident electric field $\vec{\varphi} = \hat{x} e^{i\vec{K}\cdot\vec{r}}$ for the two-space scatterer formalism scattering amplitude $g = g(\hat{r}, \hat{r}) = \hat{x} \cdot g(\hat{K}, \hat{K})$ in the forward direction. The WKB approximation (12) consists of replacing the field inside the scatterer by a field traveling in the direction of the incident excitation which propagates inside the scatterer with propagation constant of the medium of the scatterer. For the two space scatterer formalism, the propagation parameter of the incident wave is different from that of the radiated wave. Therefore, the conventional WKB method is not applicable here. Using a modified version given in (4) and the two space scatterer formalism volume integral representation of the scattered wave, it can be shown that

$$g(\hat{K}, \hat{K}) \cdot \hat{x} = \frac{i\kappa_1}{4\pi} \mathcal{T} \left[\left(1 - \frac{1}{\mu_2} \right) \kappa_1 \kappa_2 - \left(\kappa_1^2 - \frac{\kappa_2^2}{\mu_2} \right) \right] I_1 - \frac{i\kappa_1}{4\pi} \mathcal{T} \left[\left(1 - \frac{1}{\mu} \right) \kappa_1 K - \left(\kappa_1^2 - \frac{K^2}{\mu} \right) \right] I_2. \quad [5]$$

Here, as in (4) and (12),

$$\mathcal{T} = \frac{2}{1 + \frac{\kappa_2 \mu}{K \mu_2}}, \quad I_1 = \frac{2\pi}{i(\kappa_2 - \kappa_1)} \left[e^{iqa} \left(\frac{a}{iq} + \frac{1}{q^2} \right) - \frac{1}{q^2} \right]_{q_1}^{q_2}, \quad [6]$$

$$q_2 = 2\kappa_2 - K - \kappa_1, \quad q_1 = \kappa_1 - K,$$

$$I_2 = \frac{4\pi}{a(K - \kappa_1)} \left[\frac{\sin[a(K - \kappa_1)]}{a^2(K - \kappa_1)^2} - \frac{\cos[a(K - \kappa_1)]}{a(K - \kappa_1)} \right].$$

Inserting [5] into [4] yields the desired results for the bulk propagation number K

$$(K^2 - \kappa_1^2) = \rho \mathcal{T} \left[\left(1 - \frac{1}{\mu_2} \right) \kappa_1 \kappa_2 - \left(\kappa_1^2 - \frac{\kappa_2^2}{\mu_2} \right) \right] I_1 - \rho \mathcal{T} \left[\left(1 - \frac{1}{\mu} \right) \kappa_1 K - \left(\kappa_1^2 - \frac{K^2}{\mu} \right) \right] I_2 \quad [7]$$

and the bulk index of refraction η

$$(\eta^2 - 1) = \rho \mathcal{T} \left[\left(1 - \frac{1}{\mu_2} \right) \kappa_2 / \kappa_1 - \left(1 - \frac{\kappa_2^2}{\mu_2 \kappa_1^2} \right) \right] I_1 - \rho \mathcal{T} \left[\left(1 - \frac{1}{\mu} \right) \eta - \left(1 - \frac{\eta^2}{\mu} \right) \right] I_2 \quad [8]$$

where the physical constants μ_2 , and μ are the magnetic permeability of the medium inside the water droplet and the bulk magnetic permeability respectively. The formulae for the calculation of μ , and ϵ in the forward direction of scattering are obtained from (11:28)

$$\begin{aligned}\mu^{-1} &= 1 - 2\pi\rho(\mathcal{F} - \mathcal{F}')/\eta\kappa_1^2, \quad \epsilon = 1 + 2\pi\rho(\mathcal{F} + \mathcal{F}')/\kappa_1^2, \\ \mathcal{F} &= \frac{g}{i\kappa_1}, \quad \mathcal{F}' = \mathcal{F}(-\kappa_1).\end{aligned}\quad [9]$$

To work with [7-9], we need to calculate the value of $\kappa_1 = k_o\eta_1 = k_o(\eta_{1r} + i\eta_{1i})$, and $\kappa_2 = \kappa_1\eta_2 = \kappa_1(1.33 + i7.33 \times 10^{-8})$. The real part η_{1r} of η_1 is obtained from the formula given by (13)

$$\begin{aligned}(\eta_{1r} - 1)10^6 &= \left[0.378,125 + \frac{0.002,141,4}{\lambda^2} + \frac{0.000,017,93}{\lambda^4}\right] \\ &\times \frac{p\{1 + (1.049 - 0.015t)p \times 10^{-6}\}}{1 + 0.003661t} - \left[0.0624 - \frac{0.000,680}{\lambda^2}\right] \frac{f}{1 + 0.003661t}\end{aligned}\quad [10]$$

where from (14), f (the vapor pressure at $t=10^\circ C$) = 9,209 mmHg, and the pressure $p = 720$ mmHg is the lowest permissible for [10]. With $\lambda = 0.7774 \mu m$ in [10], we have $\eta_{1r} = 1.00026476$, and $\kappa_{1r} = 8.08443$.

To obtain the imaginary part κ_{1i} of κ_1 , we use the result of (15) for the attenuation of intensity in the air $Ie^{-\alpha d} = 0.8I$, where I is the intensity, $\alpha = 2\kappa_{ai}$ (the subscript a is used to indicate air), and the path is taken to be 11 km. From the above, we have $\kappa_{ai} = 1.014 \times 10^{-11}$. Using Archimedes principle, the value of κ_{ai} , and the sphericity of the water droplets in the cloud, we can write $\kappa_1 = 8.08443 + i1.0446 \times 10^{-11}$, $\kappa_2 = 10.75229 + i.59260 \times 10^{-6}$.

To proceed with these calculations, we must verify the validity of the approximation that gives [5]. The WKB approximation is valid according to (12) if the host medium is such that $\Re(\kappa_1)2a(\epsilon_{1r} - 1) \gg 1$ and $(\epsilon_{1r} - 1) < 1$ where $\epsilon_1 = \epsilon_{1r} + i\epsilon_{1i}$, $\epsilon_{1r} = \eta_{1r}^2 - \eta_{1i}^2$, $\epsilon_{1i} = 2\eta_{1r}\eta_{1i}$ since the magnetic permeability μ_1 is assumed to be one. In this case, the above conditions are satisfied. From the imaginary part of κ_1 , we deduce the value of $\eta_{1i} = (\kappa_{1i}/k_o) = 1.2924 \times 10^{-12}$.

Equations [7], [8], and [9] form a coupled nonlinear system for the bulk parameters K , η , ϵ , and μ . To solve the system numerically, we approximate the exponential factor of I_1 by its leading term and use the values given in (1) as initial data. Hence, we have

$$\begin{aligned}K &= 8.08442974208285 + i3.14117832893745 \times 10^{-8}, \\ \epsilon &= 1.00000000014550 + i3.87231835597287 \times 10^{-9}, \\ \mu &= 1.00000000014570 + i3.89603100094467 \times 10^{-9}, \\ \eta &= .999999968400000 + i3.88417455500000 \times 10^{-9}, \\ \mathcal{F} &= 15.1452985315156 + i404.033753935163, \\ \mathcal{F}' &= -1.01021251084684 \times 10^{-2} + i1.23330047641423.\end{aligned}\quad [11]$$

The results of [11] are close as expected, to those given in (1). Should the incident wave were magnetic in nature, ϵ and μ would have been interchanged. To improve [11], one

can upgrade the model and construct $f(\mathbf{R})$ numerically. The values given in [11] can be directly applied to the problem of lightning due to point sources located inside the cloud.

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